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Construction of stability regions in the three-body problem using parameter elimination☆

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ABSTRACT

It is shown, using the example of an investigation of the stability of relative equilibrium positions (libration points) in the classical three-body problem and in several modifications of it, that in many cases it, is simpler and more informative to construct the stability region of equilibrium positions in the configuration space of the system rather than in the parameter space.

The problem of the stability of steady motions can be reduced to the solution of a system of n equations in n generalized coordinates and parameters of the system. The conventional procedure consists of solving the system for the generalized coordinates, substituting the solution into inequalities that express the stability conditions and constructing the stability region in parameter space. However, this system is generally non-linear and difficult to solve analytically for the generalized coordinates. Nevertheless, certain parameters often exhibit linear behaviour in the equations, and representing them in the form of functions of the generalized coordinates is an easily solvable problem. After some of the parameters are eliminated, the stability region can be constructed in configuration space or in a mixed space of parameters and generalized coordinates rather than in parameter space. This sometimes produces an informative and physically clear picture of the stability region.

Such an approach is effective for constructing the stability regions of relative equilibrium positions (libration points) in the classical three-body problem and several of its modifications.

Consider the constant Lagrangians of the solution of the unrestricted three-body problem in a generalized Laplace formulation, in which the bodies are located at the vertices of an equilateral triangle. The conditions for orbital stability of these solutions are known in a first approximation (the necessary conditions for Routh–Zhukovskii stability[1,2])

$$\frac{m_2m_3+m_3m_1+m_1m_2}{(m_1+m_2+m_3)^2}<\frac{1}{3}\left(\frac{\alpha+3}{\alpha-1}\right)^2,\quad \alpha>-3 \tag{1}$$

where m_1, m_2 and m_3 are the masses of the bodies and α is the exponent in the power law for mutual gravitation (for Newtonian gravitation $\alpha=-2$). A physically informative picture of the stability region can be obtained in configuration space if we consider the positions of the centre of mass of the bodies instead of the values of the masses of the three points. Let r be the radius vector of the centre of mass drawn from the geometric centre of an equilateral triangle whose vertices are occupied by the bodies. The first of the inequalities in (1) is equivalent to the inequality (ρ is a side of the triangle)[3]

$$r^2>r_c^2=\frac{\rho^2}{3}\left(1-\left(\frac{\alpha+3}{\alpha-1}\right)^2\right) \tag{2}$$

When the right-hand side is positive, inequality (2) indicates that the centre of mass of the system must be located outside the circle of radius r_c with its centre at the geometric centre of the equilateral triangle formed by the point masses (Fig. 1).

We will trace the evolution of the stability region as the parameter α varies. As can be seen from relation (2), as α varies from -3 to -1, the radius r_c of the boundary of the stability region decreases from $r_c=r_{\max}=\rho/\sqrt{3}$ to 0, and the stability region expands. When $\alpha=-3$ and $r_c=r_{\max}$, the stability region vanishes, because $r_{\max}$ is equal to the radius of the circle circumscribed about the triangle, outside of

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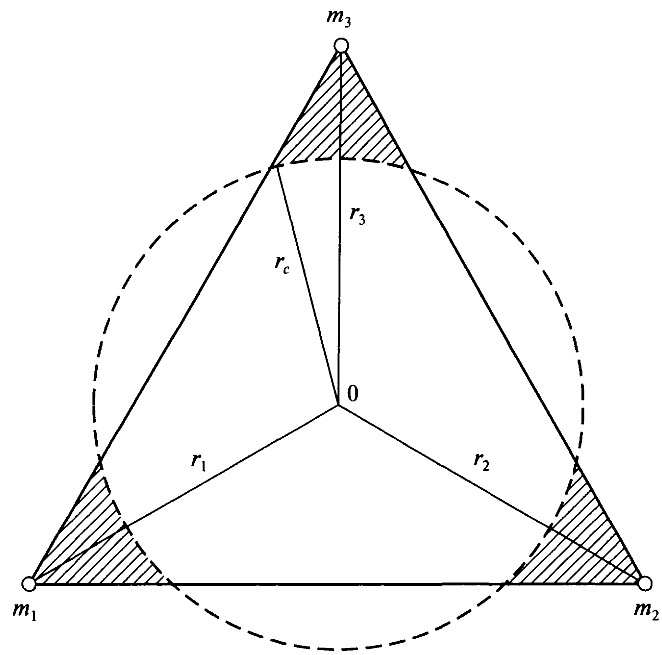

Fig. 1.

which the centre of mass of the system cannot be located. When $\alpha = -1$, the circle contracts to a point. When $\alpha > -1$, the right-hand side of inequality (2) becomes negative, and the stability conditions hold for any mass. In the case of Newtonian gravitation ($\alpha = -2$), which is of practical interest, the ratio $r_c/r_{\max} = \sim 0.94$, and the boundary of the region where the necessary conditions for stability hold closely approaches the vertices of the triangle. This leads to the well-known conclusion that stability is possible in this case only when one of the masses is large compared with the other two.

The question of stability by virtue of the complete equations of perturbed motion remained open for a long time, and it was established only comparatively recently[4,5] that in the region where the necessary conditions for stability hold, there are two resonance sets of mass values (that correspond to third- and fourth-order internal resonance conditions), under which the Lagrangians of the solutions are unstable by virtue of the complete (non-linear) equations of perturbed motion. These sets correspond to two sets of positions of the centres of mass of the system in the form of two concentric circles with the same centre O that intersect the stability regions shown in Fig. 1. In the remaining cases where the necessary conditions for stability (1) hold, the question of stability by virtue of the non-linear system has not been solved.

We now consider the restricted circular three-body problem, for which the transition to configuration space enables us to give a simple proof for the existence of three and only three collinear libration points. As we know,[6] these points are unstable; however, because they are highly attractive for dealing with circumlunar space, various methods for stabilizing them have been developed, one of which[7] will be discussed below.

The conventional approach[6] to determining the coordinates of collinear libration points involves transforming the non-linear equilibrium equation (x_1 and x_2 are the coordinates of points with masses $1-\mu$ and μ, and the distance between these masses is taken as the unit of length)

$$-x + (1-\mu)\frac{x - x_1}{|x - x_1|^3} + \mu\frac{x - x_2}{|x - x_2|^3} = 0$$

which specifies the coordinates of the collinear libration points, into a fifth-order algebraic equation in x. The coefficients in this equation depend non-linearly on the mass parameter μ, and the proof that it has three real roots, which are the coordinates of collinear libration points, is not trivial.

However, if the origin of coordinates of the system is placed not at the centre of mass of the system, but at some fixed point (independent of μ) on the straight line passing through the principal bodies, for example, at a point with mass $1-\mu$, the equilibrium equation indicated becomes linear in the parameter μ:

$$\mu\left[\frac{x}{|x|^3} - \frac{x-1}{|x-1|^3} + 1\right] = \frac{x}{|x|^3} - x$$

A graph of the function $\mu(x)$ is shown in Fig. 2 for the entire physically possible range of variation of the mass parameter μ $(0 < \mu < 1)$. The conclusion that there are three collinear points for any fixed μ becomes clear from this figure. The figure also reveals the nature of the relative positioning of the collinear libration points as μ varies.

We will now consider several modified versions of the restricted circular three-body problem. In the first of these, which is called the photogravitational three-body problem,[8] the motion of a microparticle in a repulsive gravitational field of binary star systems, which comprise a significant part of all star systems, is considered. Thus, it will be assumed that a passively gravitating particle will experience radiation pressure forces from the principal bodies S_1 and S_2 in addition to the gravitational forces. In this case, in the orbital planes of

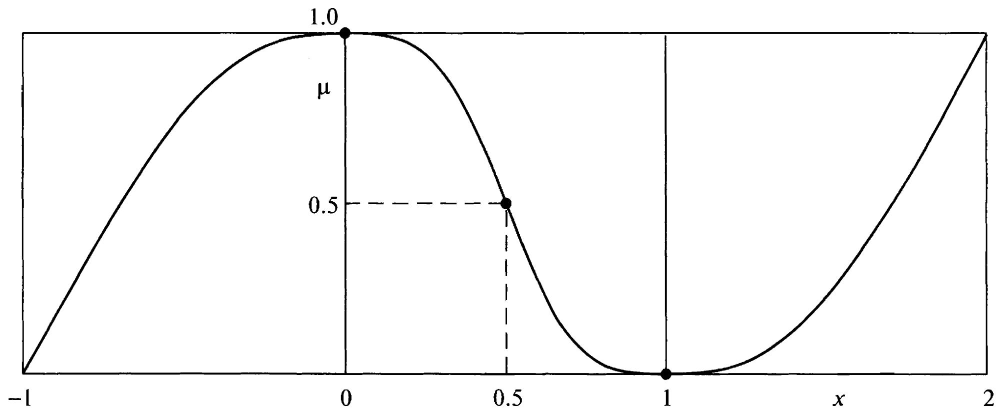

Fig. 2.

the principal bodies (the stars) there are three-parameter families of relative equilibrium positions of the microparticles.[9,10] All three parameters (the mass parameter μ, which is equal to the relative mass of one of the stars, and the reduction coefficients of the particles Q_1 and Q_2, which represent the ratios of the difference between the gravitational force and the radiation pressure force to the gravitational force) appear as linear parameters in the equations that specify the relative equilibrium positions of the particle. Solving these equations for Q_1 and Q_2, we obtain $Q_i = r_i^3$ (Refs 9 and 10), where r_i denotes the distances from the particle to each of the bodies S_1 and S_2 (the distance between the bodies is taken as the unit of length). It follows from the physical meaning of the reduction coefficients that $Q_i \le 1$, i.e., the region where the libration points exist is bounded by two circles of unit radius with centres at S_1 and S_2, which intersect at the classical libration points L_4 and L_5 (Fig. 3).

For any fixed stellar pair (S_1, S_2), the reduction coefficients are constrained by the condition[9,10] $(1-Q_2)/(1-Q_1)=k$, where k is a parameter that characterizes the repulsive gravitational field of the stellar pair (c_1 and c_2 are the optical radiation powers of the stars)

$$k = \frac{c_2/\mu}{c_1/(1-\mu)}$$

The reduction coefficients Q_1 and Q_2 also depend on the properties of the particles; therefore, in the planes of the orbits of the stellar pair there are curves (cloud clusters) that are specified by the equation

$$(1-r_2^3)/\left(1-r_1^3\right) = k$$

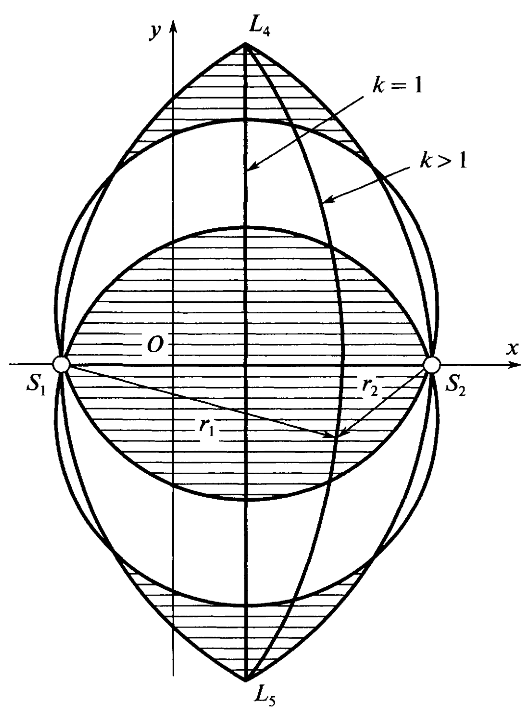

Fig. 3.

and consist of relative equilibrium positions of particles with different reduction coefficients. These curves are shown in Fig. 3 for $k = 1$ and $k > 1$.

The necessary conditions for stability of the libration points found[9,10] are obtained from the requirement that the characteristic equation of the linearized system of equations of the perturbed motion does not have any roots with positive real parts. After eliminating the parameters Q_1 and Q_2 using the formulae indicated above, which can be obtained from the equilibrium positions, these necessary conditions take the form

$$0 \le 9\mu(1-\mu)\sin^2(\psi_1+\psi_2) \le 1/4$$

where ψ_1 and ψ_2 are the angles made with the Ox axis by the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of the particle, which are drawn from the points S_1 and S_2 and will hold for all libration points if μ does not exceed the smaller root of the equation

$$36\mu(1-\mu) = 1$$

At large values of μ, the stability region allows of the following simple geometrical interpretation in the configuration space of the system (i.e., in the system of coordinates Oxy with origin at the centre of mass).[9,10] It consists of three parts (they are shown hatched in Fig. 3), whose boundaries are arcs of two circles with an identical radius that depends on μ. The circles have the segment S_1S_2 as a common chord (their centres lie on the segment L_4L_5 on opposite sides of this chord at the same distance from it). One of these parts of the stability region contains the entire segment S_1S_2 and is symmetrical about it. The other two parts (which are also symmetrical about the Ox axis) are adjacent to the classical triangular libration points L_4 and L_5 and are bounded on the right and on the left by circles of unit radius. The stability region, whose shape and dimensions are specified only by the parameter μ, is intersected by curves that correspond to different values of the parameter k and represent families of microparticles with different area/mass ratios.

One more modification of the restricted three-body problem, in which a small reactive acceleration $\mathbf{w}$ (which is constant in absolute value and has an invariant orientation in the system of coordinates $Oxyz$ rotating about the Oz axis) was imparted to a passively gravitating point for the purpose of stabilizing the positions of the collinear libration points, which, as we know, are unstable in the classical problem, was considered in Ref 7.

Under the assumptions made, the force field of the problem is still a potential field with the force function

$$U = W(x, y, z) + \mathbf{w}\cdot\mathbf{r}$$

where $W(x, y, z)$ is the force function of the classical problem and $\mathbf{r}$ is the radius vector of the passively gravitating point, drawn from the centre of mass of the system, at which the origin of the rectangular system of coordinates $Oxyz$ with the Ox axis directed along the straight line connecting the principal bodies is placed.

From the conditions for relative equilibrium of the passively gravitating point in the Oxy plane

$$\frac{\partial W}{\partial x} + w\cos\alpha = 0, \quad \frac{\partial W}{\partial y} + w\sin\alpha = O$$

where α is the angle the reactive acceleration vector makes with the Ox axis, we can easily determine the absolute value of the acceleration and the angle α

$$w = \sqrt{\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2}, \quad \mathrm{tg}a = \frac{\partial W}{\partial y}\left(\frac{\partial W}{\partial x}\right)^{-1} \tag{3}$$

while finding the coordinates of the equilibrium positions as functions of w and α is an analytically unsolvable problem.

It is seen from expression (3) that any point in the Oxy plane can be made a relative equilibrium position by selecting the appropriate value for the acceleration. The question of which of these positions will be stable is solved by investigating the equations of perturbed motion, which contain the second partial derivatives of the force function U. A characteristic feature of the problem under consideration is the fact that these derivatives do not contain the acceleration w and are identical to the second partial derivatives of the force function W of the classical problem. Therefore, when constructing the stability region, it does not matter whether the additional acceleration acts on the point or not. Only equilibrium conditions (3) should be used to select its magnitude so that it fall's within the stability region found, which is thus the invariant region of the classical restricted three-body problem.

The inequalities obtained enable us to construct a region of gyroscopic stability in the configuration space of the system, and they contain the parameter μ, which is the dimensionless mass of the small body. Because of the fairly complex dependence of the inequalities obtained on the coordinates x and y, the boundaries of the stability region specified by them can only be found numerically. Corresponding calculations for the Earth–Moon system were previously performed.[7] They showed that stability regions exist and occupy a part of the Oxy plane located near the outer collinear libration point L_3 of the system. Although these stability conditions were obtained in a first approximation, it may asserted,[11] by virtue of the Hamiltonian nature of the system, that stability is maintained when terms up to any finite high order are taken into account in the equations of perturbed motion.

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References

1. Routh E. On Laplace's three particles, with supplement on the stability of a steady motion. *Proc London Math Soc* 1875;**6**:86–97.
2. Zhukovskii NYe. *Collected Papers. Vol 1: On the Stability of Motion*. Moscow, Leningrad: Gostekhizdat; 1937. pp. 110–218.
3. Kunitsyn AL. A geometrical interpretation of the necessary conditions for stability of the triangular libration points of the general three-body problem. *Celest Mech* 1971;**3**(2):222–6.
4. Kunitsyn AL, Tkhai VN. On the instability of Laplace's solutions of the unrestricted three-body problem. *Pis'ma Astron Zh* 1977;**3**(8):376–80.
5. Ivanov AP. Investigation of the stability of the constants of the Lagrange solutions of the plane unrestricted three-body problem. *Prikl Mat Mekh* 1979;**43**(5):787–95.
6. Duboshin GN. *Celestial Mechanics. Fundamental Problem and Methods*. Moscow: Nauka; 1968.
7. Dzhumabayeva AA, Kunitsyn AL, Tuyakbayev A. On the stabilization of collinear libration points in the Earth–Moon system. *Prikl Mat Mekh* 1999;**63**(2):196–203.
8. Radziyevskii VV. The restricted three-body problem with allowance for the radiation pressure. *Astron Zh* 1950;**27**(4):249–56.
9. Kunitsyn AL, Chudayeva AM. On the stability of clusters of microparticles in a repulsive gravitational field of binary star systems. *Prikl Mat Mekh* 2003;**67**(5):731–8.
10. Kunitsyn AL, Tureshbayev AT. The stability of triangular libration points of the photogravitational three-body problem. *Pis'ma Astron Zh* 1985;**11**(2):145–8.
11. Birkhoff GD. *Dynamical Systems*. New York: American Mathematical Society; 1927.

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